The Effect of Higher-Order Corrections on the Propagation of Nonlinear Dust-Acoustic Solitary Waves in a Dusty Plasma with Nonthermal Ions Distribution

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Z. Naturforsch. **63a,** 261 – 272 (2008); received December 11, 2007

Propagation of nonlinear dust-acoustic (DA) waves in a unmagnetized collisionless mesospheric dusty plasma containing positively and negatively charged dust grains and nonthermal ion distributions are investigated. For nonlinear DA waves, a reductive perturbation method is employed to obtain a Korteweg-de Vries (KdV) equation for the first-order potential. As it is well-known, KdV equations contain the lowest-order nonlinearity and dispersion, and consequently can be adopted for only small amplitudes. As the wave amplitude increases, the width and velocity of a soliton can not be described within the framework of KdV equations. So, we extend our analysis and take higher-order nonlinear and dispersion terms into account to clarify the essential effects of higher-order corrections. Moreover, in order to study the effects of higher-order nonlinearity and dispersion on the output solution, we address an appropriate technique, namely the renormalization method.

Key words: Mesospheric Dusty Plasma; Dust-Acoustic Waves; KdV-Type Equation; Renormalization Method; Solitary Solution.

1. Introduction

Dusty plasmas are low-temperature ionized multispecies gases including electrons, ions, and negatively (or positively) charged dust grains typically with micrometer or submicrometer size. This state of plasma is ubiquitous in the universe, e. g., in interstellar clouds, in interplanetary space, in cometary tails, in ring systems of giant planets, in mesospheric noctilucent clouds, as well as in many earth-bound plasmas, see for instance [1, 2]. Two important effects arise from the presence of dust grains, viz the reduction of the number of the electrons, as some of them are adsorbed on the grains, and the introduction of a new time scale. The charge-to-mass ratios of dust particles are smaller by several orders of magnitude than those of conventional plasma particles, leading to their support of various waves of much lower frequency such as dust-acoustic (DA) waves [3, 4], dust-ion-acoustic (DIA) waves [5, 6], and dust-lattice (DL) waves [7, 8]. Therefore the study of dust plasmas has become the focus of research in the last decade. Indeed, dust grains in a plasma are not neutral, but they act as probes and collect the background plasma electrons and ions. It turned out that the dust grains immersed in a gaseous plasma are usually negatively charged. However, recently there are many direct evidences for the existence of both negatively and positively charged dust grains not only in Earth's mesosphere [9] but also in cometary tails and comae [10 – 12]. Basically, there are three principle mechanism by which dust grains become positively charged: photoemission in the presence of a flux of ultraviolet (UV) photons, thermionic emission induced by radiative heating, and secondary emission of electrons from the surface of dust grains. For example, the interaction of photons incident to the dust grain surface causes photoemission of electrons from the dust grain surface. The dust grains, which emit photoelectrons, may become positively charged. The emitted electrons collide with other dust grains and are capture by some of these grains which may become negatively charged. Also, energetic plasma particles are incident to a dust grain surface may lose their energy partially or fully. A portion of the lost energy can go into exciting other electrons that in turn may escape from the dusty surface.

From theoretical perspective, Chow et al. [11] have shown that due to the size effect on secondary emission insulating dust grains with different sizes in space plasmas can have the opposite polarity (smaller ones being positively and larger ones being negatively charged). This is mainly due to the fact that the excited secondary electrons have shorter (longer) distances to travel to reach the surface of the smaller (larger) dust grains. So it is easy to say that dust grains of different sizes in many circumstances can acquire different polarities, large grains become negatively charged and small grains positively charged. Such differences in the sign of the charge can greatly modify the properties of the plasma. Therefore, it is important and interesting to deal with dust plasma systems if, in addition to the electrons and ions, there are two types of dust particles with different masses, charges and temperatures. Consequently, there will be two basic DA modes propagating with two different velocities. It means that the linear and nonlinear waves will become interesting and may have many applications in space and astrophysical plasmas. For example, Mamun and Shukla [13] suggested a new dusty plasma model that is appropriate for cometary tails. They considered charged dust grains of opposite polarity without electrons and ions in the ambient plasma, and they found that such a two-components dusty plasma supports dust – Langmair and DA depressive waves. After that, linear and nonlinear properties of DA waves were studied in a multi-component dusty plasma model with inertialess electrons and ions as well as positively and negatively charged initial dust grains by Sakanaka and Spassovska [14]. Recently, Akhtar et al. [15] examined the propagation of DA solitary waves in a multi-components dusty plasma. In their work, two types of dust fluids (one is cold and the other is hot) have been considered in the presence of Boltzmannian ions and electrons. However, as it is well known, kinetic effects may be important in space as well as in a bounded state plasma. For instance, the velocity distribution functions of the different charged particle species, that make up a plasma, play a crucial role in influencing the linear as well as nonlinear properties of the plasma. The Maxwellian distribution function, which is in thermal equilibrium, is most frequently used in collisionless plasmas. However, various observations of fast ions and electrons in space environments indicated that these particles have velocity distributions that are not in thermal equilibrium. More specifically, nonthermal ions have been observed in the Earth's bow-shock [16]. Similarly, the loss of energetic ions from the upper ionosphere of Mars has been observed through satellite observations [17]. More recently, nonthermal ion populations which could be modelled by a Kappa distribution have been found to occur in the magnetospheres of planets like Jupiter and Saturn [18, 19]. The above observations justify the need to study the influence of nonthermal velocity distributions on waves and instabilities in space plasmas. Many authors have adopted nonthermal velocity distributions in nonlinear plasma studies. Cairns et al. [20] proposed that the nonthermal nature of electrons in a two-component model comprising Boltzmann ions as well could support the simultaneous existence of solitary waves with both positive and negative potential which could possibly account for the observations made by the Freja satellite. They found that, when they considered a thermal plasma, only a single solitary wave was supported. Mamun et al. [21] employed a model comprising nonthermal ions and negatively charged dust and reported that solitary waves with positive and negative potential can coexist. Gill and Kaur [22] studied nonlinear dust acoustic waves in a two-component plasma comprising nonthermal ions and a negatively charged dust component. They incorporated the effects of dust temperature into the model and found that the dust temperature can influence whether rarefactive solitons can simultaneously coexist with compressive solitons in the nonthermal plasma. As it is well known, the investigation of small-amplitude DA waves is usually described by the Korteweg-de Vries (KdV) equation. As a routine procedure, this equation can be derived from the basic set of fluid equations for a plasma by, for instance, the reductive perturbation theory [23]. The resulting equation contains the lowest-order nonlinearity and dispersion, and consequently can describe a wave of only small amplitude. As the wave amplitude increases, the width and velocity of a soliton deviate from the prediction of the KdV equation: the breakdown of the KdV approximation [24]. In fact, the typical solutions of the KdV equation are solitons, hump structures propagating at a given speed, but no changing forms. Such stationary (in a comoving frame), pluse-like solutions can exist because nonlinearity, leading to wave steeping, and dispersion balance each other. In order to describe DA waves of larger amplitude, higher-order dispersive and nonlinear effects must be taken into account. A stationary solution for the resulting equations has been solved via Watanabe and renormalization methods [24 – 28].

Recently, Elwakil et al. [29] examined the effect of higher-order corrections on the propagation of nonlinear DA solitary waves in mesospheric dusty plasmas. In the present work, our main concern is to examine the effect of both higher-order depressive and nonlinear corrections on the amplitude and width of DA solitary waves propagated in a bipolar three-components dusty plasma consisting of massive, micron-sized positively and negatively charged dust grains subjected to nonthermal ion distributions. This paper is organized as follows. In Section 2, we present the basic set of fluid equations governing our plasma model which reduces to the well-known KdV equation by employing the reductive perturbation theory. In Section 3, the KdV equation with the fifth-order dispersion term is presented as well as its higher-order solution. In Section 4, we extend our analysis to account the higher-order nonlinearity in order to obtain an equation for the second-order potential and then apply the renormalization method to obtain the desired solution. Finally, some discussion and conclusions are given in Sections 5 and 6.

2. Basic Equations and KdV Equation

Consider a three-component dusty plasma with massive, micron-sized, positively and negatively charged dust grains and nonthermal ion distributions. This study is based on the condition that negatively charged dust particles are much more massive than positive ones [10, 12]. In addition, most of the background electrons could be stick onto the dust grains surface during the charging processes and, as a result, one might encounter a significant depletion of electron number density in the ambient dusty plasma. Therefore, our basic governing equations read

$$\frac{\partial n}{\partial t}+\frac{\partial (nu)}{\partial x}=0, \tag{1a}$$

$$\mu\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)+\frac{\partial \phi}{\partial x}+\frac{\sigma}{n}\frac{\partial p}{\partial x}=0, \tag{1b}$$

$$\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+3p\frac{\partial u}{\partial x}=0 \tag{1c}$$

for a positively charged dust plasma and

$$\frac{\partial N}{\partial t}+\frac{\partial (NV)}{\partial x}=0, \tag{2a}$$

$$\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial x}-\frac{\partial \phi}{\partial x}=0 \tag{2b}$$

for a negatively charged dust plasma.

Equations (1) and (2) are supplemented by Poisson's equation:

$$\frac{\partial^2 \phi}{\partial x^2}=N-\mu_1 n-\mu_2 n_i. \tag{3}$$

In the above equations n and u are the density and velocity of positively charged dusty grains while N and V are the density and velocity of negatively charged dusty grains, n_i is the density of ions, ϕ and p are the electric potential of dust fluid and the thermal pressure of the positively charged dust fluid, respectively. Here n and N are normalized by their equilibrium values n_0 and N_0, u and V are normalized by $C_s=\sqrt{\mu}V_T$, $\mu=Z_\mathrm{n}m_1/Z_\mathrm{p}m_2$, $V_T=(Z_\mathrm{p}k_\mathrm{B}T_\mathrm{n}/m_1)^{\frac{1}{2}}$, $Z_\mathrm{p}(Z_\mathrm{n})$ represents the number of the positive (negative) charges on the dust grain surface, $m_1(m_2)$ represents the mass of the positive (negative) dust particle, k_B is the Boltzmann constant, T_n is the temperature of the negatively charged dust fluid, P is normalized by $n_0k_\mathrm{B}T_\mathrm{p}$, since T_p is the temperature of the positively charged dust fluid, and ϕ is normalized by $k_\mathrm{B}T_\mathrm{n}/Z_\mathrm{n}e$, x is the space variable normalized by $\lambda_{ID}=(k_\mathrm{B}T_\mathrm{n}/4\pi N_0Z_\mathrm{n}e^2)^{\frac{1}{2}}$, t is the time variable normalized by $\omega_{p2}^{-1}=(m_2/4\pi N_0Z_\mathrm{n}^2e^2)^{\frac{1}{2}}$, where $\sigma=\frac{T_\mathrm{p}}{Z_\mathrm{p}T_\mathrm{n}}$, $\mu_1=\frac{n_0Z_\mathrm{p}}{N_0Z_\mathrm{n}}$, and $\mu_2=\frac{n_{i0}}{N_0Z_\mathrm{n}}$. The nonthermal ion distributions n_i can be expressed as [30]

$$n_i=\left(1+\beta\phi+\beta\phi^2\right)\mathrm{e}^{-\phi} \tag{4a}$$

with,

$$\beta=\frac{4\delta}{1+3\delta}, \tag{4b}$$

where δ is a parameter determining the number of fast (nonthermal) ions [20]. It should be noted that if we neglect the number of nonthermal ions in comparison with that of thermal ions, i. e. we let $\delta=0$, their density is given by the Boltzmann distribution.

To derive the KdV equation describing the behaviour of the system for longer times and small but finite amplitude DA waves, we introduce the slow stretched co-ordinates

$$\tau=\varepsilon^{\frac{3}{2}}t,\ \xi=\varepsilon^{\frac{1}{2}}(x-\lambda t), \tag{5}$$

where ε is a small dimensionless expansion parameter, and λ is the speed of DA waves. All physical quantities appearing in (1) – (3) are expanded as power series in ε

about their equilibrium values as

$$\begin{aligned} n &= 1+\varepsilon n_1+\varepsilon^2 n_2+\varepsilon^3 n_3+\cdots, \\ u &= \varepsilon u_1+\varepsilon^2 u_2+\varepsilon^3 u_3+\cdots, \\ N &= 1+\varepsilon N_1+\varepsilon^2 N_2+\varepsilon^3 N_3+\cdots, \\ V &= \varepsilon V_1+\varepsilon^2 V_2+\varepsilon^3 V_3+\cdots, \\ p &= 1+\varepsilon p_1+\varepsilon^2 p_2+\varepsilon^3 p_3+\cdots, \\ \phi &= \varepsilon \phi_1+\varepsilon^2 \phi_2+\varepsilon^3 \phi_3+\cdots. \end{aligned} \tag{6}$$

We impose the boundary conditions

$$\begin{aligned} &|\xi| \to \infty, \quad n = N = 1, \quad p = 1, \\ &u = V = \phi = 0. \end{aligned} \tag{7}$$

Substituting (4) and (5) into (1)–(3), and equating coefficients of like powers of ε, from the lowest-order equations in ε, the following results are obtained:

$$\begin{aligned} &n_1 = \rho\phi_1, \quad u_1 = \lambda\rho\phi_1, \\ &p_1 = 3\rho\phi_1, \\ &N_1 = \frac{-\phi_1}{\lambda^2}, \quad V_1 = \frac{-1}{\lambda}\phi_1, \end{aligned} \tag{8a}$$

where

$$\rho = \frac{-\beta\mu_2+\mu_2+\frac{1}{\lambda^2}}{\mu_1}. \tag{8b}$$

Poisson's equation gives the linear dispersion relation

$$1+\left(3\sigma-\lambda^2\mu\right)\rho = 0. \tag{8c}$$

If we consider the coefficients of $O(\varepsilon^2)$, we obtain with the aid of (8a) the following set of equations:

$$-\lambda\frac{\partial n_2}{\partial\xi}+\frac{\partial u_2}{\partial\xi}+\rho\frac{\partial\phi_1}{\partial\tau}+(2\lambda\rho^2)\phi_1\frac{\partial\phi_1}{\partial\xi} = 0, \tag{9a}$$

$$\begin{aligned} &\lambda\mu\rho\frac{\partial\phi_1}{\partial\tau}+\sigma\frac{\partial p_2}{\partial\xi}-\lambda\mu\frac{\partial u_2}{\partial\xi} \\ &+\left(3\rho^2\sigma+\lambda^2\mu\rho^2\right)\phi_1\frac{\partial\phi_1}{\partial\xi}+\frac{\partial\phi_2}{\partial\xi} = 0, \end{aligned} \tag{9b}$$

$$3\rho\frac{\partial\phi_1}{\partial\tau}-\lambda\frac{\partial p_2}{\partial\xi}+3\frac{\partial u_2}{\partial\xi}+12\lambda\rho^2\phi_1\frac{\partial\phi_1}{\partial\xi} = 0, \tag{9c}$$

$$-\frac{1}{\lambda^2}\frac{\partial\phi_1}{\partial\tau}-\lambda\frac{\partial N_2}{\partial\xi}+\frac{\partial V_2}{\partial\xi}+\frac{2}{\lambda^3}\phi_1\frac{\partial\phi_1}{\partial\xi} = 0, \tag{10a}$$

$$-\frac{1}{\lambda}\frac{\partial\phi_1}{\partial\tau}-\lambda\frac{\partial V_2}{\partial\xi}+\frac{1}{\lambda^2}\phi_1\frac{\partial\phi_1}{\partial\xi}-\frac{\partial\phi_2}{\partial\xi} = 0, \tag{10b}$$

$$(-1+\beta)\phi_2+\frac{1}{2}\phi_1^2+\frac{\partial^2\phi_1}{\partial\xi^2}+n_2-N_2 = 0. \tag{11}$$

Eliminating the second-order perturbed quantities n_2, u_2, N_2, V_2 and ϕ_2 in (9)–(11), we obtain the following KdV equation for the first-order perturbed potential:

$$\frac{\partial\phi_1}{\partial\tau}+\mathbf{A}\phi_1\frac{\partial\phi_1}{\partial\xi}+\frac{\mathbf{B}}{2}\frac{\partial^3\phi_1}{\partial\xi^3} = 0, \tag{12}$$

where

$$\mathbf{A} = \frac{3\left(\mu\lambda^2+\sigma\right)\rho^2\mu_1\lambda^3+\left(\lambda^2\mu-3\sigma\right)\left(\lambda^3\mu_2-1\right)}{2\mu\rho\mu_1\lambda^4+(\lambda+1)(\lambda^2\mu-3\sigma)},$$

$$\mathbf{B} = \frac{\lambda^5\mu-3\lambda^3\sigma}{2\mu\rho\mu_1\lambda^4+(\lambda+1)(\lambda^2\mu-3\sigma)}.$$

A desired stationary wave solution of (12) can be obtained by introducing the transformation of the dependent variables ξ and τ to $\eta = \xi-\Omega\tau$, where Ω is a constant solitary wave velocity normalized by the dust acoustic speed C_d, and imposing the boundary conditions for localized disturbances, viz $\phi \to 0$, $\frac{\mathrm{d}\phi}{\mathrm{d}\eta} \to 0$, and $\frac{\mathrm{d}^2\phi}{\mathrm{d}\eta^2} \to 0$, at $\eta \to \pm\infty$, as in [31]:

$$\phi_0 = \phi_m \operatorname{sech}^2(D\eta), \tag{13}$$

where ϕ_m and D^{-1} are referred to the amplitude and width of the soliton, respectively, given by

$$\phi_m = \left(\frac{3\Omega}{A}\right), \quad D^{-1} = \sqrt{\frac{2B}{\Omega}}.$$

3. Higher-Order Dispersion Correction

It is clear that (12) contains the lowest-order nonlinearity and dispersion, and consequently can describe a wave of only small amplitude. As the wave amplitude increases, the width and velocity of a soliton deviate from the prediction of the KdV equation (12), i. e., the breakdown of the KdV approximation. So, to describe DA waves of large amplitude, the higher-order nonlinearity and dispersive effect have to be taken into account. In this regard, we examine first the effect of a KdV equation associated with the fifth-order dispersion method:

$$\frac{\partial\psi}{\partial\tau}+A\psi\frac{\partial\psi}{\partial\xi}+\frac{B}{2}\frac{\partial^3\psi}{\partial\xi^3}+\varepsilon\frac{\partial^5\psi}{\partial\xi^5} = 0, \tag{14}$$

where ε is a smallness parameter. If ε is zero, (14) reduces to the well-known KdV equation (12). The last term of the left-hand side in (14) is a perturbation added to the KdV equation (12) as a higher-order dispersive effect. Due to this term, we can not solve the above equation exactly and have to rely on a perturbation method. Remarkably, the higher-order equation includes secular terms. The elimination of these secular terms will modify the soliton velocity, i. e., the secularity in the higher-order equation is renormalized to the velocity.

To solve (14), substituting $\eta = \xi - \Omega\tau$, one gets

$$-\Omega\frac{d\psi}{d\eta} + A\psi\frac{d\psi}{d\eta} + \frac{B}{2}\frac{d^3\psi}{d\eta^3} + \varepsilon\frac{d^5\psi}{\partial\eta^5} = 0. \quad (15)$$

For (15), we present a simple method [24] for obtaining higher-order solitary wave solutions. According to this method, we expand $\psi(\eta)$ and Ω with respect to the smallness parameter ε as

$$\begin{aligned}\psi &= \psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \varepsilon^3\psi_3 + \ldots, \\ \Omega &= \Omega_0 + \varepsilon\Omega_1 + \varepsilon^2\Omega_2 + \varepsilon^3\Omega_3 + \ldots.\end{aligned} \quad (16)$$

Substituting (16) into (15) and collecting together the coefficient of like power in ε, one obtains the following equations:

$$\varepsilon^0 : -\Omega_0\frac{d\psi_0}{d\eta} + A\psi_0\frac{d\psi_0}{d\eta} + \frac{B}{2}\frac{d^3\psi_0}{d\eta^3} = 0, \quad (17a)$$

$$\varepsilon^1 : L\psi_1 = \Omega_1\frac{d\psi_0}{d\eta} - \frac{d^5\psi_0}{d\eta^5}, \quad (17b)$$

$$\varepsilon^2 : L\psi_2 = \Omega_1\frac{d\psi_1}{d\eta} + \Omega_2\frac{d\psi_0}{d\eta} - A\psi_1\frac{d\psi_1}{d\eta} - \frac{d^5\psi_1}{d\eta^5}, \quad (17c)$$

$$\begin{aligned}\varepsilon^3 : L\psi_3 = {} & \Omega_3\frac{d\psi_0}{d\eta} + \Omega_2\frac{d\psi_1}{d\eta} + \Omega_1\frac{d\psi_2}{d\eta} \\ & - A\frac{d(\psi_1\psi_2)}{d\eta} - \frac{d^5\psi_2}{d\eta^5},\end{aligned} \quad (17d)$$

where the operator L is represented by

$$L = -\Omega_0\frac{d}{d\eta} + A\frac{d}{d\eta}\psi_0 + \frac{B}{2}\frac{d^3}{d\eta^3}. \quad (18)$$

This operator is the one-variable version of the linearized KdV operator.

Equations (17a)–(17d), including the third-order, and the higher-order derivatives have a secular term like the higher-order equations of the reductive perturbation method, and the elimination of the secular terms determines the velocity correction. We solve (17) successively under the boundary conditions that ψ_i, $\frac{d\psi_i}{d\zeta}$, and $\frac{d^2\psi_i}{d\zeta^2}$ $(i = 0, 1, 2, \ldots)$ vanish at $\eta = \pm\infty$. Equation (17a) yields a solitary wave solution in the form

$$\psi_0 = \psi_m \operatorname{sech}^2(D\eta), \quad (19)$$

where the soliton amplitude ψ_m and the soliton width D^{-1} are given by

$$\psi_m = \frac{3\vartheta_0}{A}, \quad D^{-1} = \sqrt{\frac{2B}{\Omega_0}}.$$

The solution given by (19) is just one soliton solution of the KdV equation.

In the next order of ε, substituting (19) into (17b), one finds

$$\begin{aligned}L\psi_1 = {} & \\ & \operatorname{sech}(D\eta)^4\left(\frac{-1440D^5\Omega_0}{A} - \frac{144BD^3\Omega_0^2}{A^2}\right)\tanh(D\eta) \\ & + \operatorname{sech}(D\eta)^6\left(\frac{2160D^5\Omega_0}{A} + \frac{216BD^3\Omega_0^2}{A^2} + \frac{54D\Omega_0^3}{A^2}\right)\tanh(D\eta) \\ & + \operatorname{sech}(D\eta)^2\left(\frac{96D^5\Omega_0}{A} - \frac{6D\Omega_0\vartheta_1}{A}\right)\tanh(D\eta). \quad (20)\end{aligned}$$

Equation (20) is a third-order linear inhomogeneous ordinary equation with respect to ψ_1. The homogeneous equation $L\psi_1 = 0$, satisfying the boundary conditions, has a solution that is proportional to $\operatorname{sech}^2(D\eta)\tanh(D\eta)$. Therefore, the secular term existing previously has disappeared on the right-hand side. Let us assume the solution of the above equation in the form

$$\psi_1 = \mu_1\operatorname{sech}^2(D\eta) + \mu_2\operatorname{sech}^4(D\eta). \quad (21)$$

It is easy to observe that the coefficient of $\operatorname{sech}^2(D\eta)$ on the left-hand side cancels out and $L\psi_1$ is expressed in terms of $\operatorname{sech}^4(D\eta)$ and $\operatorname{sech}^6(D\eta)$. Then the coefficient of $\operatorname{sech}^2(D\eta)$ on the right-hand side should vanish, which leads to the correction of the velocity:

$$\Omega_1 = \frac{4\Omega_0^2}{B^2}. \quad (22)$$

The inhomogeneous equation without secularity is solved, leading to

$$\psi_1 = \frac{-30\Omega_0^2}{AB^2}\operatorname{sech}^2(D\eta) + \frac{45\Omega_0^2}{AB^2}\operatorname{sech}^4(D\eta). \quad (23)$$

The above solutions (22) and (23) agree with the solutions obtained by other perturbation methods [32].

In the next orders of ε, substituting (21), (22) and (23) into (17c) and (17d), we can follow the same procedure used before to evaluate ϑ_1 and ψ_1. The results are

$$\Omega_2 = 0, \qquad \psi_2 = \frac{90{\Omega_0}^3}{AB^4}\operatorname{sech}^2(D\eta) - \frac{1395{\Omega_0}^3}{AB^4}\operatorname{sech}^4(D\eta) + \frac{1395{\Omega_0}^3}{AB^4}\operatorname{sech}^6(D\eta), \tag{24}$$

$$\vartheta_3 = 0,$$

$$\psi_3 = \frac{-3708{\Omega_0}^4}{AB^6}\operatorname{sech}^2(D\eta) + \frac{31554{\Omega_0}^4}{AB^6}\operatorname{sech}^4(D\eta) - \frac{99324{\Omega_0}^4}{AB^6}\operatorname{sech}^6(D\eta) + \frac{74493{\Omega_0}^4}{AB^6}\operatorname{sech}^8(D\eta). \tag{25}$$

Combining (21)–(25), we obtain a solution of the KdV equation with a fifth-order dispersion term. It is worth noting that the higher-order solutions are expressed by power series of the lowest-order solution. This means that the wave velocity depends only on the first order of ε, while the wave form of a solitary wave depends on all orders of ε.

4. Higher-Order Nonlinearity Correction

In this section we intend to examine both higher-order dispersion and nonlinearity; so we start by using (9)–(12). The second-order perturbed quantities n_2, u_2, p_2, N_2 and V_2 can be written in terms of ϕ_1 and ϕ_2 as

$$n_2 = \frac{1}{2\theta_1\theta_2}\Big\{\rho\left[3\rho + \mu(3-\lambda^4+3\lambda^2\rho)\theta_2\phi_1^2 + 2\theta_1\phi_2\right] - (2\lambda^4\mu\rho\theta_2)\frac{\partial^2\phi_1}{\partial\xi^2}\Big\}, \tag{26a}$$

$$u_2 = -\frac{\rho}{4\lambda\theta_1}\{-9-3\lambda^2(\mu+6\rho) + \lambda^4(3+-2\rho\mu-3\rho^2)+\mu\lambda^6(1+\rho)\}\phi_1^2 - \frac{1}{4\lambda\theta_1\theta_2}\Big\{-4\lambda^2\theta_1\phi_2 + 2\lambda^4(-9+\lambda^4\mu^2)\rho\frac{\partial^2\phi_1}{\partial\xi^2}\Big\}, \tag{26b}$$

$$p_2 = -\frac{3\rho}{2\theta_1}\{-3\rho+\mu[3+5\lambda^2\rho+\lambda^4(-1+2\rho^2)]\}\phi_1^2 - \frac{1}{2\theta_1\theta_2}\Big\{3(-3\theta_1\phi_2 - 2\lambda^4\rho\theta_2)\frac{\partial^2\phi_1}{\partial\xi^2}\Big\}, \tag{26c}$$

$$N_2 = \frac{1}{2\theta_1}\{-3+3\rho(\mu+\rho)+\lambda^2(\mu+3\mu\rho)\}\phi_1^2 + \frac{-2\theta_1\phi_2 + 2\lambda^2\theta_2\frac{\partial^2\phi_1}{\partial\xi^2}}{2\lambda^2\theta_1}, \tag{26d}$$

$$V_2 = \frac{1}{4\lambda^3\theta_1}\{3-\lambda^2\mu+\lambda^4(-3+2\mu\rho+3\rho^2) + \lambda^6(\mu+3\mu\rho^2)\}\phi_1^2 - \frac{\phi_2}{\lambda} - \frac{\lambda\theta_2\frac{\partial^2\phi_1}{\partial\xi^2}}{2\theta_1}, \tag{26e}$$

where

$$\theta_1 = -3+\mu\lambda^2+\lambda 4\mu\rho, \quad \theta_2 = -3+\mu\lambda^2. \tag{26f}$$

If we now turn our attention to the next order of ε in (1), (2), and (3) and use (6), (7), and (8), we obtain the following set of equations:

$$\frac{\partial n_2}{\partial\tau} + u_2\frac{\partial n_1}{\partial\xi} + u_1\frac{\partial n_2}{\partial\xi} - \lambda\frac{\partial n_3}{\partial\xi} + n_2\frac{\partial u_1}{\partial\xi} + n_1\frac{\partial u_2}{\partial\xi} + \frac{\partial u_3}{\partial\xi} = 0, \tag{27a}$$

$$\frac{\mu\partial u_2}{\partial\tau} - \sigma\frac{n_1\partial p_2}{\partial\xi} + \sigma\frac{\partial p_3}{\partial\xi} + \frac{\mu u_2\partial u_1}{\partial\xi} + \frac{\mu u_1\partial u_2}{\partial\xi} - \frac{\mu\lambda\partial u_3}{\partial\xi} + \frac{\partial\phi_3}{\partial\xi} = -\frac{\sigma}{2}\frac{n_1^2\partial p_1}{\partial\xi} + \sigma\frac{n_2\partial p_1}{\partial\xi}, \tag{27b}$$

$$\frac{\partial p_2}{\partial\tau} + u_2\frac{\partial p_1}{\partial\xi} + u_1\frac{\partial p_2}{\partial\xi} - \lambda\frac{\partial p_3}{\partial\xi} + 3p_2\frac{\partial u_1}{\partial\xi} + 3p_1\frac{\partial u_2}{\partial\xi} + 3\frac{\partial u_3}{\partial\xi} = 0, \tag{27c}$$

$$\frac{\partial N_2}{\partial\tau} + V_2\frac{\partial N_1}{\partial\xi} + V_1\frac{\partial N_2}{\partial\xi} - \lambda\frac{\partial N_3}{\partial\xi} + N_2\frac{\partial V_1}{\partial\xi} + N_1\frac{\partial V_2}{\partial\xi} + \frac{\partial V_3}{\partial\xi} = 0, \tag{28a}$$

$$\frac{\partial V_2}{\partial\tau} + V_2\frac{\partial V_1}{\partial\xi} + v_1\frac{\partial V_2}{\partial\xi} - \lambda\frac{\partial V_3}{\partial\xi} - \frac{\partial\phi_3}{\partial\xi} = 0, \tag{28b}$$

$$\frac{\partial^2\phi_2}{\partial\xi^2} - N_3 + \mu_1 n_3 - \Big(\frac{1}{2}\beta\mu_2 - \frac{1}{6}\mu_2\Big)\phi_1^3 + \mu_2\phi_1\phi_2 - \mu_2\phi_3 + \beta\mu_2\phi_3 = 0. \tag{28c}$$

Eliminating n_3, u_3, N_3, V_3 and ϕ_3 from (27) and (28) and with the aid of both (8) and (12), a linear inhomogeneous equation for the second-order perturbed potential ϕ_2 can be obtained:

$$\widetilde{L}(\phi_1)\phi_2 \equiv \frac{\partial\phi_2}{\partial\tau} + A\frac{\partial(\phi_1\phi_2)}{\partial\xi} + \frac{B}{2}\frac{\partial^3\phi_2}{\partial\xi^3} - S(\phi_1) = 0, \tag{29}$$

where

$$S(\phi_1) = A_1 \frac{\partial \phi_1^3}{\partial \xi} + A_2 \Big(\frac{\partial}{\partial \xi} \phi_1 \frac{\partial^2 \phi_1}{\partial \xi^2} \Big) + A_3 \phi_1 \Big(\frac{\partial^3 \phi_1}{\partial \xi^3} \Big) - A_4 \frac{\partial^5 \phi_1}{\partial \xi^5} \quad (30)$$

and the coefficients A_i $(i = 1, 2, \ldots, 4)$ are:

$$\begin{aligned} A_1 = {} & (\chi\lambda^3)^{-1}\{(\lambda^2\mu - 3\sigma)(6\rho^5(\mu^3\lambda^6 + 2\mu^2\sigma\lambda^4 \\ & - 27\mu\sigma^2\lambda^2 + 18\sigma^3)\mu_1^3\lambda^{10} + 6\rho^4(3(\lambda+2)\mu^3\lambda^6 \\ & - (14\lambda+17)\mu^2\sigma\lambda^4 - 3(31\lambda+36)\mu\sigma^2\lambda^2 \\ & + 9(12\lambda+11)\sigma^3\mu_1^2\lambda^8 + \rho^3\mu_1(3(\lambda^2\mu - 3\sigma) \\ & \cdot (\mu^2\lambda^4 - 21\mu\sigma\lambda^2 - 114\sigma^2)(\lambda+1)^2 \\ & + 2\lambda\mu_1((19\lambda+21)\mu^3\lambda^5 - 6(10\lambda+7)\mu^2\sigma\lambda^3 \\ & + 2(\mu^3\lambda^6 + 9\mu^2\sigma\lambda^4 + 36\mu\sigma^2\lambda^2 - 54\sigma^3)\mu_2\lambda^3 \\ & - 9(15\lambda+7)\mu\sigma^2\lambda + 108\sigma^3\lambda^6 - (\rho^2(\lambda^2\mu - 3\sigma)\mu_1 \\ & - (\lambda(13\lambda+22)+21)\mu^2\lambda^4 - 2(2(2\lambda+5)\mu^2\lambda^4 \\ & + 3(7\lambda+3)\mu\sigma\lambda^2 + 9(11\lambda+9)\sigma^2\mu_2\lambda^4 \\ & + 4\mu^2\mu_1((3\beta+1)\mu_2\lambda^6 - 5\lambda - 2)\lambda^4 + 6(\lambda(14\lambda \\ & + 17) + 7)\mu\sigma\lambda^2 + 9(\lambda(29\lambda+32)+7)\sigma^2\lambda^4 \\ & + 2\rho(\lambda^2\mu - 3\sigma)\mu_1(\mu_2(-(\lambda(2(3\beta+1)\lambda(\lambda+1) \\ & - 5) - 7)\mu^2\lambda^3 + 3(\lambda(2(3\beta+1)\lambda(\lambda+1) - 1) \\ & - 7)\mu\sigma\lambda + 36\sigma^2 + (\mu^2\lambda^7 - 9\mu\sigma\lambda^5 - 18\sigma^2\lambda^3)\mu_2\lambda^3 \\ & + (\lambda(4\lambda+7)+4)\mu^2\lambda^2 - 18\sigma^2 - 3(\lambda(7)+4)\mu\sigma\lambda^4 \\ & + (\lambda^2\mu - 3\sigma)^3(\mu_2(2(\lambda+2)\mu_2\lambda^4 - (\lambda(3\beta(\lambda+1)^2 \\ & + \lambda(\lambda+2) - 2) - 6)\lambda + 7\lambda^4 + 2(\lambda^2 + \lambda + 1)6\lambda))\}, \end{aligned}$$

$$\begin{aligned} A_2 = {} & \chi^{-1}\{(\lambda(\lambda^2\mu - 3\sigma)(9\rho^3(\mu^3\lambda^6 - 4\mu^2\sigma\lambda^4 \\ & - 15\mu\sigma^2\lambda^2 - 18\sigma^3)\mu_1^2\lambda^6 + 9\rho^2(\lambda^2\mu - 3\sigma) \\ & \cdot ((\lambda+2)\mu^2\lambda^3 - 2\mu\sigma\lambda + 3\sigma^2)\mu_1\lambda^5\rho(\lambda^2\mu \\ & - 3\sigma)\mu_1((2-\lambda)\mu^2\lambda^3 + 3(7\lambda-2)\mu\sigma\lambda + 54\sigma^2 \\ & + 3(\mu^2\lambda^7 - 9\mu\sigma\lambda^5 - 18\sigma^2\lambda^3)\mu_2\lambda^3 + (\lambda^2\mu - 3\sigma)^3 \\ & \cdot (3(\lambda+2)\mu_2\lambda^4 - 2(\lambda+2)\lambda + 1))\}, \end{aligned}$$

$$\begin{aligned} A_3 = {} & (2\chi)^{-1}\{\lambda(\lambda^2\mu - 3\sigma)^2(2\rho^3(\mu^2\lambda^4 + 3\mu\sigma\lambda^2 \\ & + 18\sigma^2)\mu_1^2\lambda^6 + 2\rho^2((2\lambda+5)\mu^2\lambda^4 + 6\mu\sigma\lambda^3 \\ & + 9(4\lambda+3)\sigma^2)\mu_1\lambda^4 + 2\rho\mu_1((2\lambda+3)\mu^2\lambda^3 \\ & - 9\mu\sigma\lambda + 18\sigma^2 + (\mu^2\lambda^7 - 9\mu\sigma\lambda^5 - 18\sigma^2\lambda^3)\mu_2\lambda^3 \\ & + (\lambda^2\mu - 3\sigma)^2(2(\lambda+2)\mu_2\lambda^4 + (\lambda+2)\lambda + 3)\}, \end{aligned}$$

$$\begin{aligned} A_4 = {} & \chi^{-1}\{\lambda^5(\lambda^2\mu - 3\sigma)^2(\rho(\mu^2\lambda^4 - 9\mu\sigma\lambda^2 \\ & - 18\sigma^2)\mu_1\lambda^2 + (\lambda+2)(\lambda^2\mu - 3\sigma)^2\} \end{aligned} \quad (31)$$

with

$$\chi = 2(2\mu\rho\mu_1\lambda^4 + (\lambda+1)(\lambda^2\mu - 3\sigma))^2 \cdot ((\lambda^2\mu - 3\sigma)^2 - 3\lambda^2\sigma\mu_1).$$

In summary, we have reduced the basic set of fluid (1)–(3) to the nonlinear KdV equation (12) for ϕ_1 and the linear inhomogeneous differential equation (29) for ϕ_2, for which the source term is described by a known function ϕ_1. To obtain a stationary solution from (12) and (29), we adopt the renormalization method introduced by Kodama and Taniuti [23] and El-Labany [25] to eliminate the secular behaviour up to the second-order potential. According to this method, (12) can be added to (29) yielding

$$\widetilde{K}(\phi_1) + \sum_{n\geq 2} \varepsilon^n \widetilde{L}(\phi_1)\phi_n = \sum_{n\geq 2} \varepsilon^n S_n, \quad (32)$$

where S_2 represents the right-hand side of (29). Adding $\sum_{n\geq 1} \varepsilon^n \delta\Theta \frac{\partial \phi_n}{\partial \xi}$ to both sides of (32), where $\delta\Theta = \varepsilon\Theta_1 + \varepsilon^2\Theta_2 + \varepsilon^3\Theta_3 + \cdots$, with coefficients to be determined later, Θ_n are determined successively to cancel out the resonant term in S_n. Then, (12) and (29) may be written as

$$\frac{\partial \widetilde{\phi}_1}{\partial \tau} + A\widetilde{\phi}_1 \frac{\partial \widetilde{\phi}_1}{\partial \xi} + \frac{1}{2} B \frac{\partial^3 \widetilde{\phi}_1}{\partial \xi^3} + \delta\Theta \frac{\partial \widetilde{\phi}_1}{\partial \xi} = 0, \quad (33)$$

$$\frac{\partial \widetilde{\phi}_2}{\partial \tau} + A \frac{\partial}{\partial \xi}(\widetilde{\phi}_1 \widetilde{\phi}_2) + \frac{1}{2} B \frac{\partial^3 \widetilde{\phi}_2}{\partial \xi^3} + \delta\Theta \frac{\partial \widetilde{\phi}_2}{\partial \xi} = S_2(\widetilde{\phi}_1) + \Theta_1 \frac{\partial \widetilde{\phi}_1}{\partial \xi}. \quad (34)$$

The parameter $\delta\Theta$ in (33) and (34) can be determined from the condition that the resonant terms in $S_2(\widetilde{\phi}_1)$ may be cancelled out by the terms $\delta\Theta \partial\widetilde{\phi}_1/\partial\xi$ in (33); see for instance [25].

Let us introduce the variable

$$\eta = \xi - (\Theta + \delta\Theta)\tau, \quad (35)$$

where the parameter Θ is related to the Mach number $M = \Theta/C_\mathrm{d}$ by

$$\Theta + \delta\Theta \equiv M - 1 = \Delta M,$$

with Ω being the soliton velocity and C_d the DA velocity.

Integrating (33) and (34) with respect to the new variable η and using the appropriate vanishing boundary conditions for $\widetilde{\phi}_1(\eta)$ and $\widetilde{\phi}_2(\eta)$ and their derivatives up to second order as $|\eta| \to \infty$, one gets

$$\frac{d^2\widetilde{\phi}_1}{d\eta^2} + B^{-1}(A\widetilde{\phi}_1 - 2\Theta)\widetilde{\phi}_1 = 0, \tag{36}$$

$$\begin{aligned}&\frac{d^2\widetilde{\phi}_2}{d\eta^2} + 2B^{-1}(A\widetilde{\phi}_1 - \Theta)\widetilde{\phi}_2 \\ &= 2B^{-1}\int_{-\infty}^{\eta}\left[S_2(\widetilde{\phi}_1) + \Theta_1\frac{d\widetilde{\phi}_1}{d\eta}\right]d\eta.\end{aligned} \tag{37}$$

The one-soliton solution of (36) admits

$$\widetilde{\phi}_1 = \phi_{1m}\mathrm{sech}^2(D\eta), \tag{38}$$

where the soliton amplitude ϕ_{1m} and the soliton width D^{-1} are given by

$$\phi_{1m} = \frac{3\Theta}{A}, \quad D^{-1} = \sqrt{\frac{2B}{\Theta}}. \tag{39}$$

Substituting (38) in (36), the source term of (36) becomes

$$\begin{aligned}&2B^{-1}\int_{-\infty}^{\eta}\left[S_2(\widetilde{\phi}_1) + \Theta_1 + \frac{d\widetilde{\phi}_1}{d\eta}\right]d\eta \\ &= 2B^{-1}(\Theta_1 - 16D^4A_4)\phi_{1m}\mathrm{sech}^2(D\eta) \\ &+ \frac{2\phi_{1m}^2\Theta}{B^2}\alpha_1\mathrm{sech}^4(D\eta) + \frac{\phi_{1m}^2\Theta}{B^2}\alpha_2\mathrm{sech}^6(D\eta),\end{aligned} \tag{40}$$

with

$$\alpha_1 = 2A_2 + A_3 + \frac{10A}{B}A_4,$$
$$\alpha_2 = \frac{2B}{A}A_1 - 6A_2 - 2A_3 - \frac{20A}{B}A_4.$$

In order to cancel out the resonant terms in $S(\widetilde{\phi}_1)$, the value of $\delta\Theta$ should be

$$\Theta_1 = 16D^4A_4. \tag{41}$$

For (40) we introduce the independent variable

$$\Psi = \tanh(\eta D) \tag{42}$$

to recast (40) into

$$\begin{aligned}&\frac{d}{d\Psi}\left[(1-\Psi^2)\frac{d\widetilde{\phi}_2}{d\Psi}\right] + \left[3(3+1) - \frac{2^2}{1-\Psi^2}\right]\widetilde{\phi}_2 \\ &= \alpha_3(1-\Psi^2) + \alpha_4(1-\Psi^2)^2,\end{aligned} \tag{43}$$

where

$$\alpha_3 = \frac{4\phi_{1m}^2}{B}\alpha_1, \quad \alpha_4 = \frac{2\phi_{1m}^2}{B}\alpha_2.$$

Note that the two independent solutions of the homogeneous part of (43) can be represented in terms of the associated Legendre function of first and second kind:

$$P_3^2 = 15\Psi(1-\Psi^2), \tag{44}$$

$$\begin{aligned}Q_3^2 = &\frac{15}{2}\Psi(1-\Psi^2)\ln\left(\frac{1+\Psi}{1-\Psi}\right) \\ &+ 2(1-\Psi^2)^{-1} - 15(1-\Psi^2)^2 + 5.\end{aligned} \tag{45}$$

The complementary solution of (43) admits the form

$$\begin{aligned}\widetilde{\phi}_{2c} = &C_1(15\Psi(1-\Psi^2)) + C_2\Big(\frac{15}{2}\Psi(1-\Psi^2) \\ &\cdot\ln\left(\frac{1+\Psi}{1-\Psi}\right) + 2(1-\Psi^2)^{-1} - 15(1-\Psi^2) + 5\Big).\end{aligned} \tag{46}$$

By applying the method of variation of parameters, the particular solution of (43) take the form

$$\widetilde{\phi}_{2p} = L_1(\Psi)P_3^2(\Psi) + L_2(\Psi)Q_3^2(\Psi), \tag{47}$$

where $L_1(\Psi)$ and $L_2(\Psi)$ are given by

$$L_1(\Psi) = -\int\frac{Q_3^2(\Psi)T(\Psi)}{(1-\Psi^2)W(P_3^2(\Psi), Q_3^2(\Psi))}d\Psi,$$

$$L_2(\Psi) = \int\frac{P_3^2(\Psi)T(\Psi)}{(1-\Psi^2)W(P_3^2(\Psi), Q_3^2(\Psi))}d\mu,$$

with

$$T(\Psi) = \alpha_3(1-\Psi^2) + \alpha_4(1-\Psi^2)^2,$$

$$W(P_3^2, Q_3^2) = P_3^2\frac{dQ_3^2}{d\Psi} - Q_3^2\frac{dP_3^2}{d\Psi} = \frac{120}{1-\Psi^2}.$$

Using P_3^2 and Q_3^2, then $L_1(\Psi)$ and $L_2(\Psi)$ reduce to

$$\begin{aligned}L_1(\Psi) = &\frac{1}{32}(1-\Psi^2)^3\Big(\frac{1}{3}\alpha_3 + \frac{1}{4}\alpha_4(1-\Psi^2)\Big) \\ &\cdot\ln\left(\frac{1+\Psi}{1-\Psi}\right) + (\alpha_5)\Psi + (\alpha_6)\Psi^3 \\ &+ (\alpha_7)\Psi^5 + \frac{1}{64}\Psi^7,\end{aligned}$$

$$L_2(\Psi) = -\frac{1}{16}(1-\Psi^2)^3\Big(\frac{1}{3}\alpha_3 + \frac{1}{4}(1-\Psi^2)\alpha_4\Big)$$

with

$$\alpha_5 = \left(-\frac{1}{48}+\frac{1}{15}\right)\alpha_3 + \left(-\frac{1}{64}+\frac{1}{15}\right)\alpha_4,$$

$$\alpha_6 = \left(-\frac{1}{18}\right)\alpha_3 + \left(-\frac{33}{360}+\frac{1}{64}\right)\alpha_4,$$

$$\alpha_7 = \left(\frac{1}{48}\right)\alpha_3 + \left(-\frac{3}{320}+\frac{1}{15}\right)\alpha_4.$$

The particular solution is given by

$$\widetilde{\phi}_{2\mathrm{p}} = \frac{\phi_{1m}^2 D^2}{\Theta}(1-\Psi^2)[\alpha_8 - \alpha_2(1-\Psi^2)], \quad (48)$$

where

$$\alpha_8 = \frac{6BA_1}{A} - \frac{10}{3}A_2 - \frac{4}{3}A_3 - \frac{20A}{3B}A_4.$$

In (40) the first term is the secular one, which can be eliminated by renormalization of the amplitude. Also the boundary conditions $\widetilde{\phi}_2 = 0$ as $\eta \to \infty$ produce $C_2 = 0$. Thus only the particular solution contributes to $\widetilde{\phi}_2$.

Expressing (48) in terms of the old variable η, the solution of (43) is given by

$$\widetilde{\phi}_{2\mathrm{p}} = \frac{\phi_{1m}^2 D^2}{\Theta}\operatorname{sech}^2(D\eta)\left[\alpha_{10} + \alpha_2 \tanh^2(D\eta)\right]. \quad (49)$$

The stationary soliton solution for DA waves is given by

$$\begin{aligned}\widetilde{\phi}(\eta) &= \widetilde{\phi}_1(\eta) + \widetilde{\phi}_2(\eta)\\ &= \left(\phi_{1m} + \frac{\phi_{1m}^2 D^2}{\Omega}\alpha_{11}\right)\operatorname{sech}^2(D\eta) \quad (50)\\ &\quad - \alpha_2 \frac{\phi_{1m}^2 D^2}{\Omega}\operatorname{sech}^4(D\eta),\end{aligned}$$

where

$$\alpha_{10} = \frac{3BA_1}{A} - \frac{1}{2}A_2 + \frac{2}{3}A_3 + \frac{10A}{3B}A_4,$$

$$\alpha_{11} = \alpha_{10} + \alpha_2,$$

$$\Theta = B^2\left[\left(1 + \frac{16A_4\Delta M}{B^2}\right)^{\frac{1}{2}} - 1\right]/(8A_4).$$

The soliton width is given by

$$D^{-1} = \left(\frac{2B}{\triangle M}\right)^{\frac{1}{2}}\left(1 + \frac{2A_4 \triangle M}{B^2}\right).$$

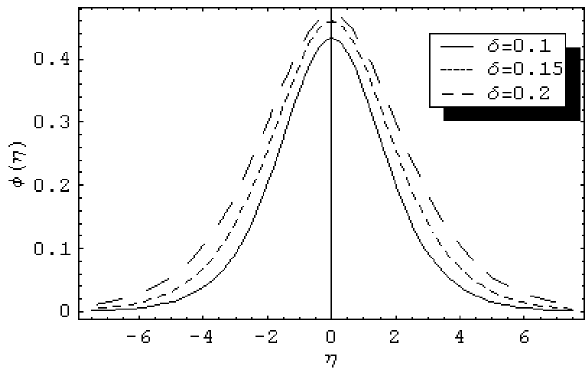

Fig. 1. Variation of the solitary wave solution $\phi_1(\eta)$ with respect to η for $\mu = 0.05$, $\mu_1 = 0.77$, $\mu_2 = 1.111$, and $\Omega = 1.6$ at $\delta = 0.1$, 0.15, and 0.2.

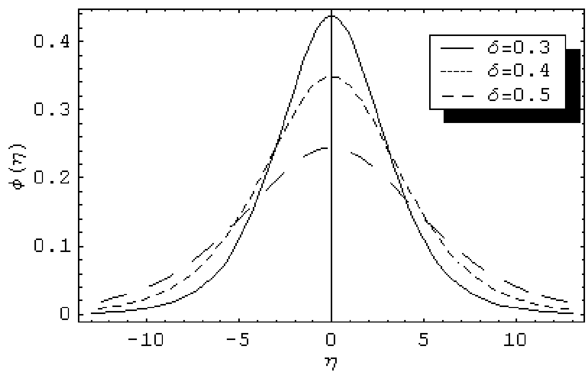

Fig. 2. Variation of the solitary wave solution $\phi_1(\eta)$ with respect to η for $\mu = 0.05$, $\mu_1 = 0.77$, $\mu_2 = 1.111$, and $\Omega = 1.6$ at $\delta = 0.3$, 0.4, and 0.5.

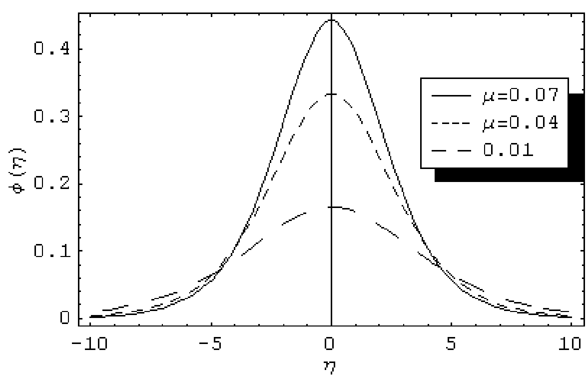

Fig. 3. Variation of $\phi_1(\eta)$ with respect to η for $\Omega = 1.6$, $\delta = 0.2$, $\mu_1 = 0.77$, and $\mu_2 = 1.111$ at $\mu = 0.01$, 0.04 and 0.07.

5. Results and Discussion

Our present dusty plasma system is unique because we have considered a charged dusty plasma of opposite

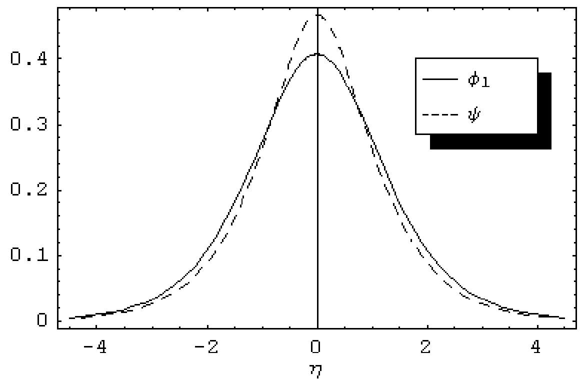

Fig. 4. Comparison between $\phi_1(\eta)$ and the higher-order dispersion contribution $\psi(\eta)$ with respect to η for $\mu = 0.07$, $\mu_1 = 0.77$, $\mu_2 = 1.111$, $\varepsilon = 0.055$, and $\Omega = 1.6$ at $\delta = 0.1$.

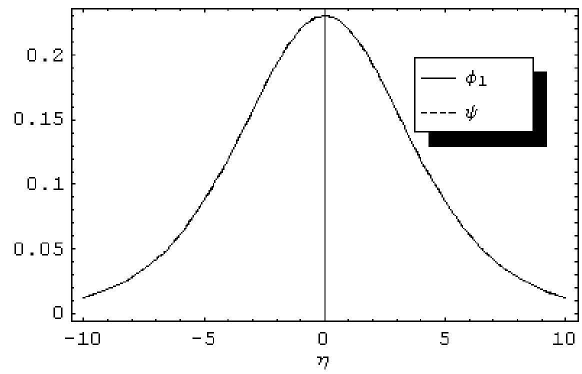

Fig. 6. Comparison between $\phi_1(\eta)$ and the higher-order dispersion contribution $\psi(\eta)$ with respect to η for $\mu = 0.07$, $\mu_1 = 0.77$, $\mu_2 = 1.111$, $\varepsilon = 0.055$, and $\Omega = 1.6$ at $\delta = 0.4$.

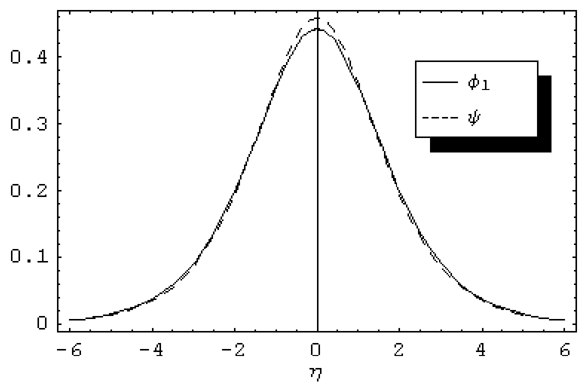

Fig. 5. Comparison between $\phi_1(\eta)$ and the higher-order dispersion contribution $\psi(\eta)$ with respect to η for $\mu = 0.07$, $\mu_1 = 0.77$, $\mu_2 = 1.111$, $\varepsilon = 0.055$, and $\Omega = 1.6$ at $\delta = 0.2$.

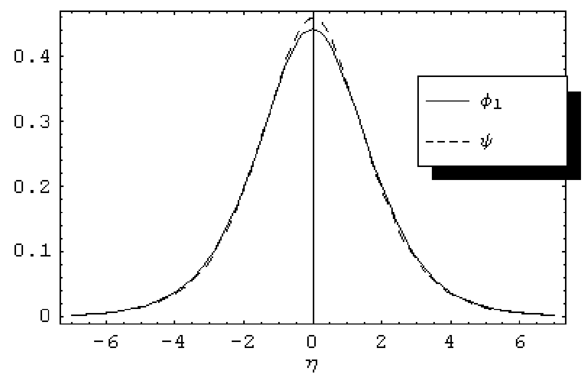

Fig. 7. Comparison between $\phi_1(\eta)$ and the higher-order dispersion contribution $\psi(\eta)$ with respect to η for $\delta = 0.2$, $\mu_1 = 0.77$, $\mu_2 = 1.111$, $\varepsilon = 0.055$, and $\Omega = 1.6$ at $\mu = 0.07$.

polarity associated with nonthermal ion distributions and without electrons in the ambient plasma. In order to make our results physically relevant, numerical studies have been made using plasma parameters close to the values in [13]. First of all, the present system supports only a compressive soliton solution as one can infer from Figure 1. Since one of our motivations was to study the effects of some plasma parameters, namely, the energetic population parameter δ and the specific charge ratio of positvely and negatively charged dusty grains μ on the formation of a solitary wave solution, i. e., the width and the amplitude, Figs. 1 – 3 are devoted to display graphically the variation between δ and μ and the characteristic features of the soliton solution. For instance, Figs. 1 and 2 show that the amplitude increases (decreases) for small (large) values of δ, while the width increases in both cases of δ. On the other hand, it is clear that the amplitude (width) increases (decreases) with increasing μ as one can infer from Figure 3.

Now, we investigated the effect of higher-order dispersion on the dust acoustic wave. In Figs. 4 – 9 the first-order approximation of the potential ϕ_1 and the higher-order dispersion correction ψ versus the amplitude η are plotted for arbitrary chosen parameters, i. e., $\delta = 0.1, 0.2, 0.4$ in Figs. 4 – 6 and $\mu = 0.07, 0.04, 0.01$ in Figs. 7 – 9, respectively. Clearly, the higher-order dispersion as shown in Figs. 4 and 7 increases the amplitude while decreases the width as expected. However, this difference decreases as depicted in Figs. 5 and 8. According to Figs. 6 and 9, there is no significant contribution for the higher-order dispersion on the output solution at the values $\delta = 0.3$ or $\mu = 0.01$. As a result, one can say that the higher-order dispersion has no rule on the output solitary waves at certain values of plasma parameters, i. e., δ (nonthermal

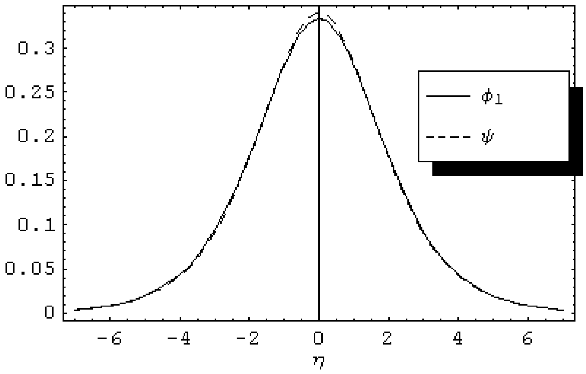

Fig. 8. Comparison between $\phi_1(\eta)$ and the higher-order dispersion contribution $\psi(\eta)$ with respect to η for $\delta = 0.2$, $\mu_1 = 0.77$, $\mu_2 = 1.111$, $\varepsilon = 0.055$, and $\Omega = 1.6$ at $\mu = 0.04$.

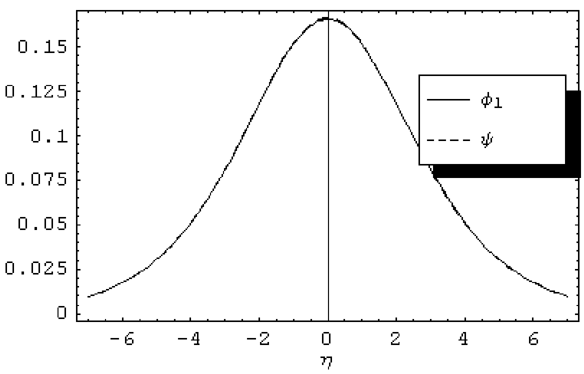

Fig. 9. Comparison between $\phi_1(\eta)$ and the higher-order dispersion contribution $\psi(\eta)$ with respect to η for $\delta = 0.2$, $\mu_1 = 0.77$, $\mu_2 = 1.111$, $\varepsilon = 0.055$, and $\Omega = 1.6$ at $\mu = 0.01$.

parameter) and μ (specific charge). Therefore, we proceeded mainly to investigate the effects of the higher-order nonlinearity in addition to the fifth-order dispersion. In order to do that, we continued our calculation to the next order of perturbed parameter ε in (1)–(3). As a result, a linearized KdV-type equation (29) with an inhomogeneous term has been derived. To obtain a stationary solution for the coupled equations (12) and (29), we have applied the renormalization method. Figures 10 and 11 show the comparison between $\psi(\eta)$ (higher-order dispersion solitary solution via the Watnabe method) versus $\widetilde{\phi}(\eta)$ (higher-order nonlinearity and dispersion solitary solution via the renormalization method) for $\delta = 0.5$ and $\mu = 0.01$, respectively. It is obvious that the higher-order nonlinearity plays an important role due to the increase of the amplitude and width of the soliton solution.

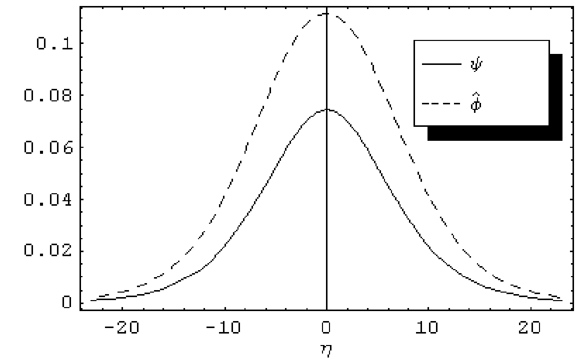

Fig. 10. Comparison between the solitary wave solution (with higher-order dispersion correction) $\psi(\eta)$ and the solitary wave solution $\tilde{\phi}(\eta)$ (with higher-order corrections), with respect to η for $\delta = 0.4$, where $\mu = 0.07$, $\mu_1 = 0.77$, $\mu_2 = 1.111$, $\varepsilon = 0.055$, and $\Omega = 1.6$.

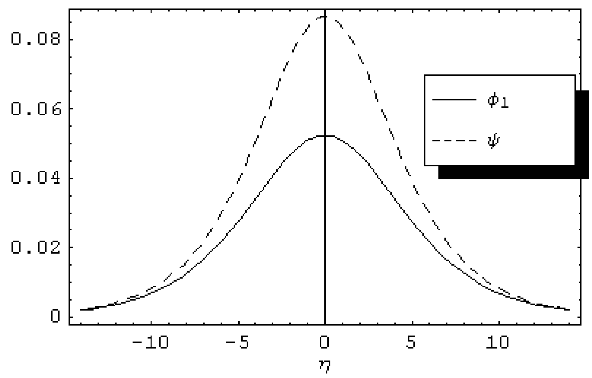

Fig. 11. Comparison between the solitary wave solution (with higher-order dispersion correction) $\psi(\eta)$ and the solitary wave solution $\tilde{\phi}(\eta)$ (with higher-order corrections) with respect to η for $\mu = 0.01$, where $\delta = 0.2$, $\mu_1 = 0.77$, $\mu_2 = 1.111$, $\varepsilon = 0.055$, and $\Omega = 1.6$.

6. Conclusion

We have devoted quite some efforts to discuss the proper description of the nonthermal energetic population δ and the specific charge ratio μ on the nature of the solitary wave solutions in unmagnetized dusty plasmas consisting of three components, namely positively, negatively charged dusty grains as well as nonthermal ion distributions. Applying the reductive perturbation theory to the basic set of fluid equations leads to the well-known KdV equation (12) which covers the propagation of nonlinear evolution of DA solitary waves. It is emphasized that the amplitude and the width of DA solitons as well as the parametric regime, where the soliton can exist, is sensitive, for example, to the energetic parameter δ. On the other hand, as is well known, as the wave amplitude increases the width and the velocity of soliton deviates from the prescribed KdV equation. Naturally, one may ask to what extent the higher-order approximations modify the soliton amplitude and the width. In other words, will the higher-order corrections increase the amplitude while decreasing the corresponding width? As we have seen, the higher-order dispersion plays an essential role due to increasing the amplitude and decreasing the width. In addition, we have found that the velocity of a soliton is mainly determined by a nonlinear term, but depends weekly on the dispersion effect. Moreover, we note that, for some values of plasma parameters, the higher-order dispersion is not effective. These results drive us to extent our analysis to account for both higher-order nonlinear and dispersion effects. A stationary solution of the resulting equation has been achieved via what is called renomalization technique. In summary, the higher-order nonlinearity plays an important role due to the increase of the amplitude while decreasing the width of the soliton solution more than one expexted. Although we have not referred to a specified situation, the present investigation would be helpful for future experiments and observations on dust acoustic waves.

Acknowledgements

We express our gratitude to the referee for a number of valuable criticisms and comments that have led to an improvement of the original paper. We also thank the editor and his staff for their kind cooperation.

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